XXI. *The Figure and Primitive Formation of the Earth, or Researches in Terrestrial Physics.*—Part I. *By* Henry Hennessy, *M.R.I.A. &c. Communicated by Major* Ludlow Beamish, *F.R.S.*

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1. The hypothesis by which the figures of the heavenly bodies are theoretically explained has long engaged the attention of geologists and geometers, without having acquired any important improvements. Great as have been the discoveries which have originated from this hypothesis, it remains in nearly the same philosophical position as that to which it had arrived when Clairaut published his Theory of the Figure of the Earth. During the period of more than a century which has elapsed since the appearance of that immortal work, the improvements which have been made in the theory of the figures of the planets, appear to consist chiefly in discoveries connected with the attractions of bodies, and in some important generalizations of the equilibrium and motions of fluids.

Comparatively few positive discoveries have however been as yet made in geology as deductions from the hypothesis of the primitive fluidity of the earth. In explanation of this circumstance several causes may be assigned, one of which appears to be the limited nature of the hypothesis. It has not been considered sufficient to suppose that the earth was originally a mass of heterogeneous matter in a fluid state; an additional supposition has been made, in support of which I am not aware that any evidence has been ever adduced. The supposition alluded to is, that the volume of the entire mass and the law of density of the fluid have not been changed by the solidification of a part of that fluid, no matter how far the solidification may have proceeded; or in other words, that the distribution of the molecules composing the earth is the same in the present state of our planet as that which they had when the mass was fluid. Although no precise evidence can be brought forward for the examination of this portion of the hypothesis, it appears not to be entirely consistent with what is known respecting the solidification of fluids. It is thus defective from appearing to involve an unproved and improbable physical law. Its exclusion from the hypothesis will make the latter more general and more applicable to the explanation of certain cosmical phenomena. Making the simplified hypothesis the basis of my investigations, it shall be their object further to generalize the theory of the figure of the earth, and to examine the nature and the energy of the physical and mechanical actions which may have been exerted upon its surface during its different geological transformations.

2. Let the earth be considered to have been originally a heterogeneous fluid mass, possessing only such general properties as those which have been established for fluids. Let x, y, z be the rectangular coordinates of μ, an elementary molecule of the fluid; and let the components of the forces, parallel respectively to the axes of x, y, z, be X, Y, Z, if the mass were not in rotation but at rest and in equilibrium. Let the mass rotate about the axis of z with an angular velocity at unity of distance represented by α, let the density at μ be represented by ϱ, a function of x, y, z; and let p represent the entire pressure upon μ, then*

$$dp=\varrho(\mathrm{X}dx+\mathrm{Y}dy+\mathrm{Z}dz)+\alpha^2(xdx+ydy).$$

The equation of the exterior surface of the fluid, and of the surfaces of equilibrium in its interior, will be

$$\mathrm{X}dx+\mathrm{Y}dy+\mathrm{Z}dz+\alpha^2(xdx+ydy)=0.$$

Let M represent the sum of the moments by which the rotation of the fluid mass was at first produced, and let the axis of z pass through the centre of gravity, then†

$$\alpha=\frac{\mathrm{M}}{\Sigma\mu(x^2+y^2)}.$$

The numerator at the right side of the above equation is evidently constant, and the denominator is a quantity depending upon the distribution of the molecules of the fluid about the centre of gravity of the entire mass. Let I represent the value of $\Sigma\mu(x^2+y^2)$ when α was the angular velocity of rotation of the body, and let I$'$ be its value corresponding to an angular velocity of rotation denoted by α'. If α should not be known independently, but if I, I$'$ and α' be known, then the equation to the free surface of the fluid can be transformed into

$$\mathrm{X}dx+\mathrm{Y}dy+\mathrm{Z}dz+\frac{\alpha'^2\mathrm{I}'^2}{\mathrm{I}^2}(xdx+ydy)=0. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (1.)$$

In any solutions heretofore given of the problem of the determination of the figure of the earth, the coefficient $\frac{\mathrm{I}'^2}{\mathrm{I}^2}$ seems not to have been noticed from the nature of the assumption which has been mentioned in (Art. 1.).

When X, Y, Z are the components produced by the mutual attractions of the molecules of the fluid according to a certain function of x, y, z, then the values of X, Y, Z will in general depend upon the form of the entire mass and of its surfaces of equilibrium, and again that form will depend upon the values of the components and of α'. As the value of I depends upon the form of the mass, it must also depend upon X, Y, Z; but it may be considered independently if the conditions upon which any particular solution of equation (1.) is to be made are such as to assign what function it may be of the coordinates. Considered under this point of view, the three first terms, and the fourth or last term at the left side of the equation (1.), may be considered as independent quantities whose values maybe found separately, and $\frac{\mathrm{I}'^2}{\mathrm{I}^2}$ may

* POISSON, Mécanique, tom. ii. p. 536, 2ieme edit. † Ibid. p. 79.

be regarded as a coefficient affecting α'^2 of such a nature as to satisfy the equation

$$\alpha=\alpha'\frac{I'}{I}. \qquad (2.)$$

3. In order to arrive at a solution of the problem of the figure of the earth, it has been found necessary to suppose that the primitive fluid mass deviated but little from the spherical form. If the spheroid be in rotation, it must follow that its axis of rotation should be that one of all the axes passing through the centre of gravity, with respect to which the sum of the forces acting upon the system would be a maximum. This axis may therefore be supposed to coincide very nearly with the position of the present axis. If no other forces be supposed to act upon the mass but the mutual attractions of its molecules and centrifugal force, then equation (1.) will rigorously represent the general equation of its surface, and the consideration of its physical or mechanical changes could be treated with less difficulty.

Whatever may be the cause of the primitive fluidity of the earth, and whatever may be the nature of the process of solidification by which at least a portion of it has arrived at a solid state, if that process should be such that a shell would be first formed surrounding a nucleus of matter in a state of fluidity, and if solidification be accompanied by a change in density of the matter solidified, it appears that some change would be produced in the motions of the entire mass about its axis, and in the arrangement and composition of that portion of the fluid constituting the nucleus. The manner in which, from physical or chemical causes, the solidifying process may proceed, will to a great extent determine what may be the actions mutually exercised by the nucleus and shell. The combination of such actions, with the changes which may have occurred in the velocity of rotation of the mass, will be thus known, and their influence upon the geological phenomena occurring during the different epochs of the physical history of the earth can be thus appreciated. In this way it may be possible to carry our ideas back to the remoter periods of the earth's existence, and to assist in explaining the phenomena which geological observers are constantly unfolding.

The application of physical and mechanical science to questions connected with geology, has been made only within a comparatively recent period, and the subject seems yet to require a more complete systematizing in order to bring it into the class of sciences which belong properly to what may be called the mechanics of the universe. An important step towards connecting geology with physical astronomy has already been made by Mr. HOPKINS in his researches concerning the constitution and thickness of the present solid shell of the globe. One of the objects of the present memoir will be fulfilled if it should in any way tend to increase that connexion.

4. Adopting in its greatest generality the hypothesis that the earth was originally a fluid mass in rotation, the particles of which attracted each other with forces varying inversely as the squares of their distances, and that the figure of the mass was nearly spherical, it will be impossible to proceed with the general investigation of the

figure which the forces acting on its different parts would ultimately produce without any knowledge of the peculiar physical nature of the fluid. It appears to be sufficiently demonstrated, that the figure which such a fluid mass would assume in virtue of the mutual attractions of its particles and of centrifugal force would be an ellipsoid of revolution. It is superfluous therefore to enter upon that portion of the subject. Assuming then the figure of the mass to be an ellipsoid of revolution, I shall endeavour to obtain general expressions for its ellipticity, and for the variation of gravity at its surface.

From what has been stated in the last paragraph of art. 2, it appears to be unnecessary to transform equation (1.) into the equation of a similar form which the conditions of the present problem would require, and from this cause the whole investigation may be considerably abridged.

Let the origin of the coordinates be at the centre of gravity of the fluid mass, let r represent the radius vector drawn from this origin to any point in any surface of equilibrium of the fluid, let θ represent the angle formed between this radius and the axis of z or axis of rotation, let ω represent the angle formed by the plane which passes through the radius r and the axis of rotation with the plane of zx, then

$$z=r\cos\theta, \quad x=r\sin\theta\cos\omega, \quad y=r\sin\theta\sin\omega.$$

Let J represent in this case the fourth term at the left side of the integral of (1.), and let f represent the centrifugal force of a point situated at distance unity from the axis, hence

$$J=\frac{1}{2}f(x^2+y^2)=\frac{1}{3}fr^2-\frac{1}{2}fr^2\left(\cos^2\theta-\frac{1}{3}\right).$$

If it be assumed that the polar and equatorial axes of the spheroid always differ but very little, we shall have for the determination at any period of f_1 the centrifugal force at the equator,

$$f_1=f'\frac{a_1 I'^2}{a' I^2}, \quad \ldots \ldots \ldots \quad (3.)$$

where a_1 represents the semipolar axis at that period, and a' the same semiaxis at present, f' being the centrifugal force at the equator at present. Let ϱ represent the density at any point of a surface of equilibrium, the polar radius of which is a, and the ellipticity e, then as the figures of the mass and of its surfaces of equilibrium differ so little from the spherical form, r may be supposed nearly equal to a, and hence for the determination of e we shall have the equation*

$$U^{(i)}\left\{\frac{4}{5}\pi a^5\int\varrho de-\frac{4}{3}\pi a^2 e\int\varrho d.a^3+\frac{4}{3}\pi\int\varrho d(a^5 e)\right\}-\frac{fa^5}{2\beta}\left(\mu^2-\frac{1}{3}\right)=0\,;$$

where $U^{(i)}$ is a function of the coordinates, βa constant, and $\mu=\cos\theta$. If E represent the ellipticity, and ϱ' the density at the exterior surface, it follows that

$$E=\frac{f_1 E}{\frac{8}{5}\pi(5E\int\varrho a^2 da+\int a^5 E d\varrho-E\varrho')}. \quad \ldots \ldots \ldots \quad (4.)$$

* Mécanique Céleste, tom. ii. p. 88.

If m' represent the ratio of centrifugal force f' to gravity g' at the equator, then

$$f_1 = \frac{m'g'a_1 I'^2}{a' I^2}.$$

If this value of f_1 be substituted in (4.), and if all the integrations be performed according to the given value of the function ϱ, an equation may be obtained after due reduction of the form

$$E = \frac{a_1 m' g' I'^2}{a'} \varphi(a), \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (5.)$$

in which equation I is considered as a function of a, and where φ is a functional symbol. That I may be considered as a function of a, is evident from the transformation of x, y, z into polar coordinates, and from the assumed identity of r and a. The above equation appears therefore to be general, or applicable to the consideration of the figure of any fluid mass upon the conditions which have formed the groundwork for its investigation.

In applying equation (5.) to the case of the earth's ellipticity, the values of m', g', a' and I', known at the present day, may be introduced. If it be supposed that the present ellipticity of the earth is the same as that which it had at the time when the first external coat of the shell had solidified from the fluid, I' will be a function of E, which can be then found if I' should be introduced under such a form by some further transformations.

The quantities I and I' are evidently the moments of inertia of the mass with respect to the axis of z during certain states of the arrangement of its molecules about the centre of gravity. Hence from the properties of ellipsoids of revolution

$$I' = \frac{2M_1 a'^2}{5\sigma},$$

σ being a number depending upon the arrangement of the particles in the interior of the mass, M_1 representing the earth's mass, and a' the present polar radius of the earth, which, without sensible error, may be used for the equatorial radius. From the consideration of the phenomena of precession and nutation, the following expression for σ' has been obtained*—

$$\sigma = \frac{P}{E - \frac{1}{2}m'},$$

where P represents a number deduced from the phenomena alluded to. If the above values of I' and σ be substituted in (5.), it will be transformed into

$$E = mQ\frac{\left(E - \frac{1}{2}m'\right)^2}{P^2},$$

* PONTÉCOULANT, Théorie Analytique du Système du Monde, tome ii. p. 475.

where Q is used for brevity instead of

$$\frac{4}{25}a_1 g' M_1^2 a'^3 \phi(a).$$

The above expression gives

$$E=\frac{1}{2mQ}\left\{m^2Q+P^2+p\sqrt{2m^2Q+P^2}\right\} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (6.)$$

If from any cause, E_1, the present ellipticity of the earth, should not be identical with E, and that we should wish to obtain the latter, the numerical value of σ, calculated from the formula

$$\sigma=P\left(E_1-\frac{1}{2}m\right)^{-1},$$

must be introduced in the value of I' and thence in equation (5.).

5. The variation of gravity at the surface of the spheroid, during its fluid state and during the process of its solidification, may now be considered, the mass being supposed in both cases to consist of spheroidal shells having a density increasing from the external surface to the centre.

From the supposition which has been just made, if γ represent the entire force of gravity at the surface of the spheroid, we shall have

$$\gamma=\frac{4\pi}{3r^2}.\int \varrho d.a^3+\frac{4\beta\pi}{r^2}\int \varrho d.\left(a^3Y_0+\frac{3a^5}{5r^2}Y_2\right)-J,$$

Y_0 and Y_2 being rational functions of the coordinates; and let it be remembered that at the surface

$$r=a_1[1+\beta(Y_0+Y_2)],$$

and that

$$\frac{4\beta\pi}{5}\int \varrho.d.a^5Y_2=\frac{4\beta\pi}{3}Y_2\int \varrho.da^3+\frac{1}{2}f\left(\cos^2\theta-\frac{1}{3}\right).$$

Let for brevity

$$G=\frac{4\pi}{a_1^2}\int \varrho a^2da-\frac{8\beta\pi Y_0}{a_1^2}\int \varrho a^2da+\frac{4\beta\pi}{a_1^2}\int \varrho da^3Y_0-\frac{2}{3}f.$$

Let G' represent the attraction at the surface of the earth at present, then if all quantities of the order β^2 and $\beta f d_1$ be neglected, we shall have after a few reductions,

$$\gamma=G+\left(\frac{5}{2}mG'\frac{I'^2a_1}{I^2a'}-EG\right)\left(\cos^2\theta-\frac{1}{3}\right);$$

$m'G'\frac{I'^2a_1}{I^2a'}$, the value of f, being here substituted for it.

If after the solidification of the external portion of the fluid into a spheroidal shell the axes of the shell undergo no great changes in magnitude, then whatever may be the physical changes occurring within the exterior surface of the shell, it will follow, from the little difference between the figure of the mass and that of a sphere, that the value of G will be invariable. Hence we may make $G'=G$, and $a_1=a'$, therefore

$$\gamma=G'\left[1+\left(\frac{5}{2}m'\frac{I'^2}{I_1^2}-E\right)\left(\cos^2\theta-\frac{1}{3}\right)\right];$$

I_1 representing the value of $\Sigma\mu\ (x^2+y^2)$ at any time after the formation of the shell. Let

$$\Gamma=G'-\frac{1}{3}G'\left(\frac{5}{2}m'\frac{I'^2}{I_1^2}-E\right);$$

then neglecting all powers of the ellipticity higher than the first,

$$\gamma=\Gamma\left[1+\left(\frac{5}{2}m'\frac{I'^2}{I_1^2}-E\right)\cos^2\theta\right]. \quad \ldots \ldots \ldots \quad (7.)$$

If $\theta=90°$, then $\gamma=\Gamma$, so that Γ will be the expression for gravity at the equator. The expression for Γ_1, gravity at the poles, will be

$$\Gamma_1=\Gamma\left[1+\left(\frac{5}{2}m'\frac{I'^2}{I_2^1}-E\right)\right].$$

Hence if we designate by S the excess of gravity at the pole above gravity at the equator divided by the latter quantity, we shall have the general theorem

$$S+E=\frac{5}{2}m'\frac{I'^2}{I_1^2}. \quad \ldots \ldots \ldots \quad (8.)$$

If $I'=I_1$, this expression will be identical with what is generally called CLAIRAUT'S theorem.

If the earth had been originally a homogeneous fluid mass, and if in solidifying it continued so, then $E=\frac{5}{4}m'$, $S=E$.

If the spheroid were heterogeneous, the excess of gravity at the pole above gravity at the equator at any period of time divided by the latter quantity, and the excess of the equatorial axis above the polar axis divided by the latter axis, would form two fractions, the sum of which multiplied by the square of the ratio of the moment of inertia of the mass at the same period to its present moment of inertia, would form a constant product which would be equal to twice the ellipticity which the spheroid would have in the case of homogeneity.

This and the foregoing article appear to contain all that it may be at present possible to deduce from the improved hypothesis of the primitive state of the earth, with relation to its figure, and to the variation of gravity at its surface. It may be hereafter possible to assign to the quantities entering $\varphi\ (a)$ in equation (5.) such values deduced from considering the physical properties of the primitive fluid, as would enable us to find a numerical value for the ellipticity.

6. In art. 3, a short general sketch has been given of the consequences which may result from the improved hypothesis of the primitive fluidity of the earth to physical geology, or to the changes occurring upon the external crust of the earth during

the process of its solidification. It is now proposed to commence the investigation of these geological consequences.

Before proceeding to the purely mathematical portions of this investigation, it seems necessary that some preliminary remarks should be made on the nature of the solidification of the globe. It seems the more necessary that these remarks should be now given, as they will not probably be found to agree entirely with those which have been made on the same subject by an eminent geologist and geometer*. It is, however, satisfactory to reflect that even if my views respecting the solidification of the earth should be more correct than those of Mr. HOPKINS, the real value of that gentleman's researches will still remain unaltered.

Although until now no allusion has been made to the probable cause of the primitive fluidity of the earth, it becomes necessary in order to proceed with the geological portion of these investigations that this cause should be assigned. Intense heat is that which is generally supposed to be the cause, and which the advocates of the earth's central heat seem to support by numerous facts. The solidification of this heated mass of fluid is so connected with its refrigeration, that some remarks on the latter subject must accompany or precede any investigations about the former.

If the whole earth from the action of a high temperature were reduced to that state of fluidity which forms the groundwork of the hypothesis upon which the theory of its figure is founded, it appears probable that the fluid mass would be extremely heterogeneous chemically, as well as mechanically. New chemical combinations would be formed, of whose physical properties it would be difficult to form an accurate idea. After the most energetic combining tendencies of the different substances composing the fluid would have been satisfied, the mass may be supposed to have arrived at a state of comparative chemical stability. Such properties of the different compounds as density and compressibility, would then exert a direct influence upon the nature of their arrangement around the centre of gravity of the spheroid. Every portion of the fluid, denser than the stratum, in which from any cause it may happen to be, would sink until it would meet a stratum of equal density. The light portions of the fluid which may happen to be in heavier strata than themselves, would on the contrary ascend until equilibrium would be obtained. The fluid mass would thus arrive at length at such a state, that it would consist of a series of spheroidal strata, each of uniform density throughout its own mass, and having that density expressible as a function of its axes.

At some time after the arrival of the fluid at this state, and perhaps before it, portions of its heat would radiate in space. The exterior portions of the fluid would cool first, until they would acquire, according to the particular circumstances which may influence their cooling, certain densities. If the effect of refrigeraion be in general an increase in density of the matter cooled, then the cooled portions of the fluid will sink. The higher temperature of the matter yet unexposed to cooling influences,

* HOPKINS, Philosophical Transactions, 1839.

would then tend to change the acquired densities of the cooled matter. The as yet uncooled fluid would have its temperature reduced from contact with the cold particles from above, and it would tend to change its position in a similar manner to the portions at first cooled. As each cooled portion of the fluid descends, the three following causes would therefore impede its descent.

1. From the arrangement already indicated of the denser strata about the centre of the mass, and from the nature of the law of density of the strata, each stratum into which the cooled fluid would descend, would be denser than the preceding.

2. If the general effect of the refrigeration be to increase the density of the cooled matter, each stratum would have its density augmented by the passage through it of the cooler matter from above.

3. The descending portions will have their densities diminished by the increase in their temperature.

During this process of circulation the interior matter in a state of fusion will be losing some of its heat by conduction. It must be evident however that if the temperature of the fluid be very high, the cooling would be carried on at first chiefly by the process of circulation. If, as analogy seems to point out, the conducting power of the fluid were small, the process of circulation would long continue to perform a principal part in the refrigeration of the fluid.

Although from the imperfect state of our knowledge with respect to the cooling of fluids it would be impossible to arrive at the precise laws and nature of the process of circulation, it yet seems probable from the three impeding causes which have been adduced, that the energy of this process would diminish from the surface of the spheroid to its centre. *The principal oscillations of the fluid produced by this cause would therefore be at first confined to the vicinity of the surface.* A solid crust may therefore be formed at the surface, long before the process of circulation could have extended to any great depths. We should thus have an external solid crust with strata of the imperfect fluid which may have been subjected to the process of circulation below it, and in the centre a mass of matter still retaining a high state of fluidity. The existence at the centre of the spheroid of a solid nucleus of small mass compared to the entire mass of the earth, cannot, as will hereafter appear, exercise any important influence on geological phenomena. If from compression, or any other cause, this solid nucleus should exist, it is evident that solidification would proceed much more slowly upon it upwards, than upon the interior surface of the shell downwards; or in other words, the increase of its mass after its first formation would be small compared to the increase of the mass of the shell subsequent to its first solidification. Whether such an internal solid nucleus exists or not, certain phenomena will attend the formation of the shell which it may be important to remark. From what is known respecting the solidification of fused substances, it may be inferred that at the moment when a portion of the fluid assumes the solid state a considerable quantity of latent heat will be eliminated. A portion of this heat will pass upwards through the shell, and another portion will pass into the imperfect fluid immediately below, tending to bring it back to a state of more perfect fluidity. If the quantity of latent heat thus eliminated should be great, it may exert an influence in changing the arrangement of the parts of the shell which it may be hereafter useful to consider. The effect of the elimination of latent heat upon the process of circulation would be evidently such as to render that process still more complex than what it was before, and its examination would thus be attended with still more difficulty and uncertainty. A very accurate examination of this portion of the phenomena attending the earth's refrigeration, appears not to be necessary for the present, and it will be found most convenient to defer its consideration, if it should be required in any future portion of these researches.

7. The motions of the shell and internal fluid matter, and their mutual actions depend so much upon the arrangement of the molecules of the primitive fluid mass, that it would be desirable before attempting their investigation to form if possible a few precise ideas respecting the last mentioned subject. The results which may be thus obtained with respect to the dimensions and law of density of the primitive fluid, will also enable us to find the probable values for $\phi\ (a)$, or Q, which should be substituted in this case respectively in either equations (5.) or (6.), in order to obtain the ellipticity of the spheroid.

Let the mass be supposed to have arrived at the state alluded to in the foregoing article, when "it would consist of a series of spheroidal strata, each of uniform density throughout its own mass, and having that density expressible as a function of its axes." Each stratum would have to sustain the normal pressures of all the strata above it, whether these pressures be produced by the action of gravity or other causes. The compressive force exercised by the upper strata upon those farther from the surface, would tend to increase the density of the latter. How far this increase of density from compression may proceed without solidification, we have no precise experimental evidence by which to judge. The celebrated experiments of Sir James Hall appear, however, to show that from the action of a great compressive force, matter in a fused state may very nearly retain the volume, and therefore the specific gravity which it had when solid. If from the combined action of increased pressure and increased temperature the specific gravity of the matter should be augmented, while its fluidity should be retained, it would fulfil the conditions of the hypothesis made in explanation of the earth's figure. If it can be assumed that the substances composing those portions of the globe inaccessible to observers are similar in constitution to those which are found at the surface, this portion of the subject appears to be susceptible of experimental examination. Until such an examination shall have been made, the mass may be supposed to have an arrangement such as that indicated in art. 6. If a solid nucleus of comparatively small mass should be formed, the densities of its strata may be supposed such, as to be expressible according to the same function of their axes as the densities of the fluid strata.

Let the fluid mass be conceived to consist of an infinite number of pyramids or columns meeting at its centre. Let, in conformity with the supposition mentioned by LAPLACE, the relation between the pressure Π at any point in one of these fluid columns, and the density ϱ be expressed by the equation*

$$\frac{d\Pi}{d\rho}=2k\varrho,$$

$2k$ being a constant depending on the peculiar physical properties of the fluid. The value of ϱ deduced from this equation, may after a few reductions be expressed under the form

$$\varrho=\frac{A}{a}\sin an;$$

where a is the semipolar axis of the spheroidal stratum in which the point may be, A a constant, and n a constant depending on the constitution of the fluid, for n is here used to express $\sqrt{\frac{2\pi}{k}}$.

Let M_1, as before, represent the mass of the spheroid which is supposed to remain constant, let ϱ' represent its density at the surface when fluid, and let D represent its mean density, so that we shall have very nearly

$$D=\frac{3M_1}{4\pi a_1^3},$$

a_1 representing the semipolar axis of the fluid mass, and π, as in every other part of this memoir, representing the ratio of the diameter of a circle to its circumference. Let c represent the amount of compression of the fluid produced by the pressure of a column l units in height, and the ratio of which to the earth's radius is represented by m_1. Then the values of n and a_1 can be determined from the equations

$$\frac{na_1}{\tan na_1}=1-\frac{c}{m};\quad \frac{D}{\varrho'}=\frac{3c}{mn^2a_1^2}.$$

Remembering the value of D, and that $m_1=\frac{l}{a_1}$, we can obtain from the second of the above equations,

$$a_1=\frac{n}{2}\sqrt{\frac{M_1 l}{c\pi\varrho'}}. \qquad (9.)$$

Let for brevity the quantity under the radical be called h^2. Then for the determination of n, we shall have the equation

$$1-\frac{cnh}{2l}-\frac{n^2h}{2\tan\frac{1}{2}n^2h}=0. \qquad (10.)$$

* Méc. Cél., tome v. p. 49.

The value of n found from this expression by successive approximations, may be substituted in (9.), whence the value of a_1 can be obtained.

By means of the operations above indicated, if sufficient data existed, some knowledge could be obtained of the arrangement of the molecules of the primitive fluid about their centre of gravity. At present no experimental data appear to exist by which numerical values could be assigned to c or l. The experiments of MM. COLLADON and STURM show that the compressibilities of different liquids vary considerably, according to their physical properties. It would be impossible therefore, from analogy with their experiments, or those of M. ŒRSTED, to form any accurate idea of the compressibility of the fluid constituting the earth when in its supposed state of fusion. The calculation of a precise value for n or a_1 seems then to be at present impossible, but geological considerations appear to impose certain limits on the value which a_1 may have had. The value of a_1 may have been equal to, or greater or less than the present semipolar axis of the earth. If it were greater, the difference should be small compared to its whole length; for if this difference were great, the external shell first formed should (no matter from what cause) arrive at its present dimensions, and in doing so it would have to undergo immense contortions of whose existence no evidence can be adduced. If a surface be conceived to pass through the centres of gravity of the infinite number of pyramidal segments into which we may conceive the shell to be divided, it is evident that all of these segments could not descend below that surface, or could not all approach in the same degree towards the centre of the mass. The shell should be broken in many places to permit the displacement of the segments, as the amount of common displacement from the contraction of the shell by refrigeration would not effect the reduction of its dimensions to those which it has at present. The irregularities of the surface of the fractured shell would evidently be in proportion to its primitive axes.

If a_1 were less than the present semipolar axis of the earth, then in order that the external coat of the shell first formed should arrive at its present dimensions, it should receive additions in some parts. If the difference between a_1 and the present semipolar axis should have been great, the effect of the contraction of the shell from refrigeration may, as in the foregoing paragraph, be neglected. From some cause, no matter what, the earth's mass should expand, and in the expansion rents would be produced in the shell. These empty spaces may be afterwards filled up with injected matter pushed forward by further expansion during the subsequent formation of the under strata of the shell. The entire shell formed of the cemented portions of the first solidified mass, and the new coats of matter added from within, may be again ruptured, and a process similar to the first may be again repeated. As the whole shell would increase in thickness, the extent and frequency of these ruptures would probably lessen, as whatever may be the mechanical action producing them, it would then have an increased resistance to overcome. The ejected matter which would arrive at the surface would also probably not amalgamate so completely with the perfectly cooled solidified matter there, as the matter formerly ejected did when the solid surface was itself at a very high temperature. From the foregoing considerations, it appears that a value of a_1 less than the present semipolar axis of the earth would be at least as much in accordance with observed geological phenomena as a value greater than that semiaxis. The protrusion of the primary and igneous rocks through the strata composing the external crust of the earth, would be in some measure a necessary consequence of the smaller value of a_1; and it would not follow that the surface of the earth should necessarily undergo any great contortions.

In either of the cases which have been just examined, the arrangement of the molecules constituting the fluid globe should be different from their present arrangement. Their arrangement may or may not be different, if the value of a_1 coincided with the present semipolar axis of the earth, in so far as the value of a_1 would have any influence upon that arrangement.

If the dimensions of the globe had been increasing in the manner which should happen if a_1 had at first a small value, the rate of that increase could be determined by finding the areas of the fissures through which the fused matter may have been ejected during the progress of the different geological formations. The geometrical relations between the surface of the spheroid and its axes would evidently be of some assistance in the course of such an investigation. If the sum of the areas of these fissures was small, compared with that of the surface of the earth, then it may be generally concluded that a_1 could not be much less than the present semipolar axis of the terrestrial spheroid.

The three following general conclusions may be now stated as deductions from the foregoing remarks:—

1. If a_1 were greater than the present semipolar axis of the earth, the difference should be small.

2. If a_1 were less than that semiaxis, it becomes more difficult to assign a limit to their difference, but it appears probable that this difference would be greater than what it would be in the foregoing case.

3. The value of a_1 may be identical with that of the present semipolar axis without contradicting any properties of the primitive fluid mass of which we may be aware, or without discordance with geological phenomena.

In order that the value of a_1 should be greater than the present semipolar axis, it is evident that the fluid should have either a low density or a small compressibility. If a_1 were, on the contrary, much less than that semiaxis, the specific gravity of the fluid should be high or its compressibility should be great. If a_1 were equal to the semipolar axis, the fluid's compressibility should still be small compared with those of all the fluids, except mercury, upon which experiments have been made, unless its density were much less than the mean density of the present external crust of the earth.

The above considerations, in addition to what have been previously presented, appear to prove that a_1 could not much exceed the present semipolar axis of the earth, and that if a_1 differed in any way from that semiaxis the difference would most probably be owing to the comparative smallness of a_1.

8. The application of equations (5.) or (6.) to the determinations of the earth's ellipticity, may now be attempted before proceeding to any further investigations. In making use of (6.) or any other similar equation for the determination of the ellipticity of the spheroid, the numerical results should be always considered as hypothetical. Such an equation could not give the present ellipticity, unless that which the earth had immediately after the solidification of the outer coat of the shell had remained unaltered. It is not improbable that many causes may have existed to produce such an alteration.

If from any cause the fluid nucleus exerted unequal pressures upon different parts of the shell, it is evident that their tendency would be to alter its form. It is possible to conceive that any alteration so produced in the figure of the earth may consist in a reduction as well as in an increase of its ellipticity. In a more advanced stage of these researches I may be able to fully consider the influence of such causes. It is enough for the present to point out the possibility of their existence.

If it be admitted that the causes which have been referred to in the foregoing paragraphs are of sufficient importance to produce appreciable changes in the general figure of the earth, it must appear at least a doubtful assumption to assign to n such a value in the expression for the density of the strata of the primitive fluid, as that which would give precisely the same ellipticity as the present.

In order to illustrate the application of equation (6.) and to show that from its form a change in the value of n can make but a comparatively small change in the value of E, I shall now proceed to find an expression for Q.

If the dimensions of the fluid mass be supposed to be the same as those of the earth at present, then g' will be the same as g, gravity at the period of its fluid state.

It is evident that $\varphi(a)$ can be expressed under the form

$$\varphi(a)=\frac{\psi(a)}{\mathrm{I}^2},$$

where

$$\mathrm{I}=\frac{8}{3}\pi\int_0^{a_1}\varrho a^4 da.$$

When a_1 is taken as unity, we shall have

$$\psi(a)=\frac{\frac{5}{2}\left(1-\frac{3q}{n^2}\right)}{\left(3-q-\frac{n^2}{q}\right)},$$

where for brevity q is used instead of $1-\frac{n}{\tan n}$; but when $\varrho=\frac{\mathrm{A}}{a}\sin an$,

$$\int_0^1\varrho a^4 da=\frac{\mathrm{A}}{n^4}[(n^2-6)q+2n^2a^2],$$

but

$$A=\frac{\varrho'}{\sin n'}, \quad D=\frac{3q\varrho'}{n^2}, \quad M_1=\frac{4}{3}\pi D,$$

hence

$$I=\frac{2}{3}\frac{M_1}{qn^2}[(n^2-6)q+2n^2];$$

whence, finally,

$$\varphi(a)=\frac{45}{8M_1^2}\cdot\frac{q^2\left(1-\frac{3q}{n^2}\right)}{\left\{2\left(1-\frac{3q}{n^2}\right)+q\right\}^2\left(3-q-\frac{n^2}{q}\right)}.$$

If this expression be written in the form

$$\frac{45}{8}\frac{K}{M^2}=\varphi(a),$$

then $Q=\frac{9}{10}K$, and (5.) will become

$$E=\frac{9}{10}\frac{m'K}{\sigma'^2}. \quad \ldots\ldots\ldots \quad (11.)$$

In making use of either (6.) or (11.), a quantity P is introduced which depends upon certain numbers obtained from observation. One of these numbers is the ratio of the earth's mass to that of the moon, and some uncertainty seems yet to exist as to its value*. The results obtained from using in (6.) or (11.) the different values of this number, which may be obtained by different methods, are much affected by their differences.

In order to compare equation (6.) with that used in the Mécanique Céleste for finding the earth's ellipticity, the following calculations have been made.

If the equation in the Mécanique Céleste be used, we shall have

$$\begin{cases} n=\frac{5}{6}\pi, & E=\frac{1}{305} \\ n=\frac{11}{12}\pi, & E=\frac{1}{336}. \end{cases}$$

If equation (6.) be used,

$$\begin{cases} n=\frac{5}{6}\pi, & E=\frac{1}{308} \\ n=\frac{11}{12}\pi, & E=\frac{1}{317}. \end{cases}$$

The value of P which has been used in the application of (6.) is that which is given in equation (*k*.), Livre V. of the work of M. De Pontécoulant.

9. After the formation of the external shell of the earth, the entire mass composed of that shell and of the internal fluid nucleus, may be supposed to rotate about an axis not differing much in position from the present terrestrial axis of rotation. If the angular velocities of the shell and nucleus about this axis were from any cause

* Pontécoulant, Théorie, &c., tom. iv., Note 3, p. 651.

different at first, it seems probable that the influence of friction, and of the cohesion of their surfaces of contact, would at length establish an uniform motion for the whole mass. The direct influence which the rotatory motion of the shell and nucleus may exercise upon geological phenomena, will not require therefore the motions of both to be separately considered.

The examination of the motions of the shell and nucleus, and of their mutual actions, whether directly, or from the influence of exterior disturbing causes, may therefore be comprehended under the following divisions:—

1. The phenomena attending the rotation of the entire mass during the gradual solidification of the shell.

2. The pressures exerted by the fluid upon the interior surface of the shell, whether from molecular action, the agency of the heavenly bodies, or from centrifugal force.

3. The influence of the changes of temperature which may occur during the solidification of the shell in contracting or expanding its parts, and in producing physical changes in its interior structure.

From the intimate connexion between the first and second of the above divisions, it would be impossible to examine one of them in a perfectly general manner without introducing the consideration of the other. The complete consideration of both could with great advantage be treated in another memoir than in the present, and I hope soon to have it in my power to complete all the necessary investigations. It will be enough for the present to indicate the importance of the first division in a geological point of view, by referring to one of the phenomena comprehended under it. If from the formation of the solid shell, and from the gradual diminution in the mass of the fluid nucleus, any change should be produced in the moment of inertia of the whole mass with respect to its axis of rotation, the angular velocity of the globe about its axis would be changed. The manner in which the solidification of the mass may take place, and the probable constitution of the fluid, will determine how far such a change may extend. If such a change should be great, it would exert an important influence upon the motions of the liquids or gases surrounding the earth, and upon a multitude of organic phenomena which its surface may have presented at different times.